Renormalization Analysis of Resonance Structure in 2-D Symplectic Map

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Abstract

A symplecticity-preserving RG analysis is carried out to study a resonance structure near an elliptic fixed point of a proto-type symplectic map in two dimensions. Through analyzing fixed points of a reduced RG map, a topology of the resonance structure such as a chain of resonant islands can be determined analytically. An application of this analysis to the Hénon map is also presented.

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1 Introduction

Since the renormalization group (RG) method was developed as an asymtotic method to study a long-time behaviour of a flow in a dynamical system, [1] there have been several attempts to extend the method to apply to discrete systems. [5, 3] Particularly, the application to a symplectic map is of great importance since a symplectic map is generally derived as a Poincare map of a Hamiltonian flow describing a physical system. However, a naive application of the RG method fails to describe a long-time behaviour of a system since the naive renormalization process does not secure the symplectic smmmetry to an RG map [4] . This difficulty generally arises in a discrete system but does not occur in the application to a Hamiltonian flow. Therefore, it is urgent to develop the RG method for a discrete system which preseves the symplectic symmetry. After some attempts to treat special cases [4],[6], a general procedure has been proposed recently in order to derive a symplecticity-preserving (symplectic) RG map [10]. The newly proposed procedure is based on the observation that a naive RG map is not symplectic but a finite truncation of the so-called Loiville expansion of a Hamiltonian flow. Then, a symplectic RG map is constructed by means of a symplectic integrator of the underlying Hamiltonian flow.

Employing the newly proposed simplecticity-preserving procedure in the RG method, we analyze resonance structures such as a chain of resonant islands near an elliptic fixed point in a two-dimensional symplectic map. A chain of resonant islands is characterized by alternating positioning of hyperbolic and elliptic periodic points in the phase space of a symplectic map. In our analysis, these periodic points are shown to be correspond to the same number of fixed points of an RG map, which simplifies the analysis to prove existence of a chain of resonant islands. As an application of the present analysis, resonance structures of the Hénon map [7] are reproduced by means of symplectic RG maps. A similar analysis to the Hénon map was tried using the RG method [8]. However, the previous study did not succeed in reconstruction of resonance structures of the Hénon map in terms of a symplectic RG map, which would be the most impressive result of an application of the RG method. Here, as an example of general results , we carry out a comprehensive study of resonance structures of the Hénon map within the framework of the RG method.

2 Symplectic Map near Elliptic Fixed point

Let us start with the following symplectic map of action-angle type , $(x_n, y_n) \mapsto (x_{n+1}, y_{n+1})$:

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1} \\ y_{n+1} &= y_n + f(x_n). \end{aligned} \tag{1}$$

The origin $(x_n, y_n) = (0, 0)$ is assumed to be an elliptic fixed point of the map (1). Expanding $f(x_n)$ around the fixed point, we have

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1} \\ y_{n+1} &= y_n + \sum_{m=1} a_m x_n^m, \end{aligned} \tag{2}$$

where a_m are constants. Since the origin is supposed to be elliptic, there are two eigenvalues at the origin: $\exp(\pm i\omega)$, where ω is real and given in terms of the coefficient a_1 as

$$2\cos\omega = 2 + a_1, \tag{3}$$

where $0 \geq a_1 \geq -4$. Eliminating y_n from Eq.(2) , we get

$$x_{n+1} - 2\cos\omega x_n + x_{n-1} = a_2 x_n^2 + a_3 x_n^3 + \cdots. \tag{4}$$

In the case $a_m = 0$ for $m > 2$, the map (4) is called the Hénon map [7]. If only the coefficient a_3 is retained while all other nonlinear coefficients vanish, Eq.(4) becomes the double-well type map studied in [4], [10]. The resonance structure near an elliptic fixed point is studied by taking Eq.(4) as a prototype in the following sections.

3 Perturbative Rrenormalization Analysis of Resonance Structure

Resonance structures in the phase plane of the prototype map (4)are analyzed by the perturbative RG method. We focus our attention to the case where the frequency ω is close to one of resonant frequencies $2\pi/k, k = 3, 4, 5$:

$$\omega = 2\pi/k + \varepsilon^p \delta, \tag{5}$$

in which ε ia a small parameter representing small resonance detuning and $p(>0)$ will be chosen later so that the small resonamce detuning balances with a nonlinear term of the leading order. Then, the map (4) reads

$$x_{n+1}-2\cos(2\pi/k)x_n+x_{n-1} = 2\{\cos\omega-\cos(2\pi/k)\}x_n+a_2x_n^2+a_3x_n^3+\cdots, \quad (6)$$

where the first term of the RHS of Eq.(6) represents small resonance detuning.

Near the origin of the phase plane, assume x_n be expanded as

$$x_n = \varepsilon x_n^{(1)} + \varepsilon^2 x_n^{(2)} + \varepsilon^3 x_n^{(3)} + \cdots, \quad (7)$$

then the first order perturbation equation gives

$$Lx_n^{(1)} \equiv x_{n+1}^{(1)} - 2x_n^{(1)}\cos(2\pi/k) + x_{n-1}^{(1)} = 0, \quad (8)$$

from which we have

$$x_n^{(1)} = A\exp\{i(2\pi/k)n\} + \bar{A}\exp\{-i(2\pi/k)n\}, \quad (9)$$

where A is a complex constant and $\bar{A}$ is the complex conjugate to A. The ordinary perturbation analysis provides $x_n^{(m)} \quad (m \geq 2)$ in terms of polynomials of A and $\bar{A}$. Among various polynomials, secular polynomials, of which coefficients depend on some powers of n, play a crucial role for a long-time behaviour of the system. In our RG method, a special transformation for A called an RG transformation is introduced [2],[4],[9] and an RG map is constructed for the renormalized A so that such secularity is removed. Let us estimate the order of magnitude of such secular polynomials. We generally encounter the following type of secular polynomial known as a term of nonlinear frequency shift.

$$(C_1 n|A|^2 + (C_2 n + C_3 n^2)|A|^4 + \cdots)A\exp(i\omega n), \quad (10)$$

where $C_m (m = 1, 2, 3)$ are constants. Thus, the term of nonlinear frequency shift is of the order ε^3 or higher.

Since the frequency of the leading order solution (9) is the resonant frequency $2\pi/k$, the following secular term appears, in addition to the nonlinear frequency shift term (10),

$$C_4 n\bar{A}^{(k-1)}, \quad (11)$$

where C_4 is a constant and k corresponds to the resonant frequency $2\pi/k$. In this paper, we call the secular term (11) a resonant secular term. The order of the resonant secular term is $\varepsilon^{(k-1)}$. Since $\bar{A}$ is the coefficient of the first order solution $\bar{A}\exp\{-i(2\pi/k)n\}$ and

$$
\begin{aligned}
[\bar{A}\exp\{-i(2\pi/k)n\}]^{(k-1)} &= \bar{A}^{(k-1)}\exp\{-i(2\pi(k-1)/k)n\} \\
&= \bar{A}^{(k-1)}\exp\{i(2\pi/k)n\},
\end{aligned}
$$

the $(k-1)$ -th order nonlinear term $[\bar{A}\exp\{-i(2\pi/k)n\}]^{(k-1)}$ causes a secular solution. For $k > 4$, the term of nonlinear frequency shift (10) is a dominant nonlinear term while the resonant secular term (11) yields small but important contribution . Since the resonance detuning $\varepsilon^p\delta$ should balance with the dominant nonlinear term, p should be chosen as $p = 2$ in this case and for $k = 4$. In the case $k = 3$, the resonant secular term dominates over the the term of nonlinear frequency shift and p should be chosen as $p = 1$ so that the resonant secular term balances with the resonance detuning term.
In the following subsections, we carry out the perturbative RG analysis of (6) for resonant frequencies $2\pi/k(k = 3, 4, 5)$.

3.1 Resonant Frequency : $2\pi/3$

For $k = 3$, the resonant frequency becomes $2\pi/3$ and the small resonance detuning is chosen to be of the order ε as

$$\omega - \omega_0 = \varepsilon\delta,$$

where $\omega_0 = 2\pi/3$. Substituting the expansion (7), where the first order solution $x_n^{(1)}$ is given by Eq.(9), into the map (6), we have the following second order equation.

$$Lx_n^{(2)} = -2\delta\sin\omega_0\{A\exp(i\omega_0 n) + \text{c.c.}\} + a_2\{A^2\exp(2i\omega_0 n) + \text{c.c.} + 2|A|^2\}, \tag{12}$$

where c.c. stands for the complex conjugate to the preceding term(s). Since $\exp(\pm 2i\omega_0) = \exp(\mp i\omega_0)$ for $\omega_0 = 2\pi/3$, the nonlinear terms proportional to A^2 and $\bar{A}^2$ and the linear detuning term in Eq.(12) causes secular terms in the second order solution $x_n^{(2)}$:

$$x_n^{(2)} = i(\delta A - \frac{a_2}{2\sin\omega_0}\bar{A}^2)n\exp(i\omega_0 n) + \text{c.c.} + a_{2,0}|A|^2, \tag{13}$$

where

$$a_{2,0} = \frac{a_2}{1 - \cos\omega_0}. \tag{14}$$

Up to the second order in ε, we have

$$x_n/\varepsilon = \Big(A + i\varepsilon(\delta A - \alpha \bar{A}^2)n\Big)\exp(i\omega_0 n) + \text{c.c.} + \varepsilon a_{2,0}|A|^2, \tag{15}$$

where

$$\alpha = \frac{a_2}{2\sin\omega_0}.$$

In order to remove secularity in Eq.(15) and derive a symplecticity-preserving RG map, we follow the procedure established in [10] or [6]. First, we introduce the renormalization transformation $A \mapsto A_n$:

$$A_n = A + i\varepsilon(\delta A - \frac{a_2}{2\sin\omega_0}\bar{A}^2)n, \tag{16}$$

from which we obtain a naive RG map $A_n \mapsto A_{n+1}$.

$$A_{n+1} = A_n + i\varepsilon(\delta A_n - \alpha \bar{A}_n^2) + \mathcal{O}(\varepsilon^2). \tag{17}$$

Then, from Eq.(15), the original variable x_n is rewritten in terms of the renormalized amplitude A_n as

$$x_n/\varepsilon = A_n \exp(i\omega_0 n) + \text{c.c.} + \varepsilon a_{2,0}|A_n|^2. \tag{18}$$

The naive RG map (17) is not symplectic but is made symplectic by means of the symplectic integration method. It should be noted that the map (17) is the first order truncation of the following Liouville expansion of a Hamiltonian flow

$$A(t+\varepsilon) = \exp(\varepsilon\mathcal{L}_H)A(t) = \Big(1 + \varepsilon\mathcal{L}_H + \frac{\varepsilon^2}{2!}\mathcal{L}_H^2 + \cdots\Big)A(t), \tag{19}$$

with a Hamiltonian

$$H(A(t), \bar{A}(t)) = i(\delta A(t)\bar{A}(t) - \frac{a_2}{6\sin\omega_0}(A(t)^3 + \bar{A}(t)^3)). \tag{20}$$

where $A(t+\varepsilon) = A_{n+1}, A(t) = A_n$ and the Liouville operator $\mathcal{L}_H$ is defined, in terms of the Poisson bracket $\{,\}$, as

$$\begin{aligned} \mathcal{L}_H A &\equiv \{A, H\}, \\ \mathcal{L}_H^2 A &= \mathcal{L}_H\Big(\mathcal{L}_H A\Big) = \{\{A, H\}, H\}. \end{aligned}$$

Therefore, the Hamiltonian (20) describes the phase space structure of the renormalized amplitude $(A_n, \bar{A}_n)$ within the order of the present approximation, regardless of choise of symplectic integrators to the Hamiltonian H. The most important points in the the phase space $(A_n, \bar{A}_n)$ are fixed points since they ,via Eq.(18), correspond to periodic points in the original phase space (x_n, x_{n+1}), which determine resonance structure such as a chain of resonant islands . The fixed points to the flow of H are solutions of the following algebraic equations

$$\frac{\partial H(A_n, \bar{A}_n)}{\partial A_n} = 0, \qquad \frac{\partial H(A_n, \bar{A}_n)}{\partial \bar{A}_n} = 0. \tag{21}$$

In the present case, Eqs.(21) are reduced to

$$\delta - \frac{a_2}{2\sin\omega_0} r \exp(-3i\theta) = 0, \tag{22}$$

where $A_n = r\exp(i\theta)(r > 0)$ and we obtain three hyperbolic fixed points

$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \qquad r = \frac{2\delta\sin\omega_0}{a_2} \tag{23}$$

for $\delta/a_2 > 0$ or

$$\theta = \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \qquad r = \frac{-2\delta\sin\omega_0}{a_2} \tag{24}$$

for $\delta/a_2 < 0$. These fixed points are also those of the naive RG map (17) and symplectic RG maps derived later. Thus, we find that the resonance structure in the original phase plane is characterized by the three hyperbolic periodic points with a period 3 instead of a chain of resonant islands. Higher order corrections to the dominant Hamiltonian (20) slightly modify the values of fixed points and do not change this result as a whole. Two symplectic RG maps are derived from the naive RG map (17) by choosing approriate canonical variables to the Hamiltonian. For canonical varibles (A'_n, A''_n) where $A_n = A'_n + iA''_n, \quad A'_n, A''_n \in \mathrm{R}$, we have

$$\begin{aligned} A'_{n+1} &= A'_n + \varepsilon f_3(A'_{n+1}, A''_n), \\ A''_{n+1} &= A''_n + \varepsilon g_3(A'_{n+1}, A''_n), \end{aligned} \tag{25}$$

where $f_3(A', A'') = -(\delta A'' + 2\alpha A'A'')$ and $g_3(A', A'') = \delta A' - \alpha((A')^2 - (A'')^2)$. Introducing the action-angle variables (J_n, θ_n) where $A_n = \sqrt{J_n}\exp i\theta_n$, we

have another symplectic RG map.

$$
\begin{aligned}
J_{n+1} &= J_n + \varepsilon F_3(J_{n+1}, \theta_n), \\
\theta_{n+1} &= \theta_n + \varepsilon G_3(J_{n+1}, \theta_n),
\end{aligned} \tag{26}
$$

where $F_3(J, \theta) = -2\alpha J^{3/2} \sin(3\theta)$ and $G_3(J, \theta) = \delta - \alpha\sqrt{J} \cos(3\theta)$. The former symplectic RG map (25) takes a set of explicit diference equations, while the latter map (26) exactly preserves a symmetry of $2\pi/3$ rotation which the Hamiltonian (20) and the naive RG map (17) possess. The RG map (26) in the angle-action variables was first derived in [8].

3.2 Resonant Frequency : $2\pi/4$

For $k = 4$, the resonant frequency ω_0 becomes $\pi/2$ and the small resonance detuning is chosen to be of the order ε^2 as

$$
\omega - \omega_0 = \varepsilon^2 \delta. \tag{27}
$$

Substituting the expansion (7) into the map (6) and taking Eq.(27) into account, we have a similar expression to the second order equation as Eq.(12), in which the frequency detuning term shoud be moved to the third order equation. Then, in contrast to the previous case, this second order equation does not include secular terms and we have the second order solution as a non-secular solution

$$
x_n^{(2)} = a_{2,2} A^2 \exp(2i\omega_0 n) + \text{c.c.} + a_{2,0}|A|^2, \tag{28}
$$

where

$$
a_{2,2} = \frac{a_2}{2(\cos(2\omega_0) - \cos\omega_0)}. \tag{29}
$$

The third order equation is written as

$$
\begin{aligned}
Lx_n^{(3)} &= \Big(-2\delta \sin\omega_0 + (2a_2(a_{2,2} + a_{2,0}) + 3a_3)|A|^2\Big) A \exp(i\omega_0 n) \\
&\quad + (2a_2 a_{2,2} + a_3) A^3 \exp(3i\omega_0 n) + \text{c.c.}.
\end{aligned} \tag{30}
$$

Since $\exp(\pm 3i\omega_0) = \exp(\mp i\omega_0)$ for $\omega_0 = \pi/2$, all forcing terms in Eq.(30) contribute to a secular solution. The third order secular solution is given as

$$
x_n^{(3)} = i((\delta + a_{3,1}|A|^2)A + a_{3,-3}\bar{A}^3) n \exp(i\omega_0 n) + \text{c.c.}, \tag{31}
$$

where

$$
\begin{aligned}
a_{3,1} &= -\frac{1}{2\sin\omega_0}(2a_2(a_{2,2}+a_{2,0})+3a_3), \\
a_{3,-3} &= -\frac{1}{2\sin\omega_0}(2a_2a_{2,2}+a_3).
\end{aligned}
$$

The secularity in $x_n^{(3)}$ is removed by the following renormalization transformation:

$$A_n = A + i\varepsilon^2((\delta + a_{3,1}|A|^2)A + a_{3,-3}\bar{A}^3\Big)n, \tag{32}$$

which yields a naive RG map.

$$A_{n+1} = A_n + i\varepsilon^2((\delta + a_{3,1}|A_n|^2)A_n + a_{3,-3}\bar{A}_n^3\Big). \tag{33}$$

The original variable x_n is represented in terms of the renormalized amplitude A_n as

$$x_n/\varepsilon = A_n\exp(i\omega_0 n) + \varepsilon a_{2,2}A_n^2\exp(2i\omega_0 n) + \text{c.c.} + \varepsilon a_{2,0}|A_n|^2. \tag{34}$$

The naive RG map (33) is derived from the Liouville expansion with a Hamiltonian

$$H = i(\delta|A_n|^2 + \frac{a_{3,1}}{2}|A_n|^4 + \frac{a_{3,-3}}{4}(A_n^4 + \bar{A}_n^4)). \tag{35}$$

Fixed points in this Hamiltonian flow, which are also fixed points of the naive RG map (33) , are determined by the similar algebraic equations as those in the previous subsection. The fixed points $A_n = r\exp(i\theta)$ satisfy

$$
\begin{aligned}
\sin(4\theta) &= 0, && (36) \\
r^2 &= -\frac{\delta}{a_{3,1}+a_{3,-3}\cos(4\theta)} > 0, && (37)
\end{aligned}
$$

which give eight fixed points

$$
\begin{aligned}
r^2 &= \frac{\delta}{2a_3}, \qquad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, && (38) \\
r^2 &= \frac{\delta}{a_2^2+a_3}, \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \pi, \frac{7\pi}{4}, && (39)
\end{aligned}
$$

where the following conditions are assumed

$$\frac{\delta}{2a_3} > 0, \qquad \frac{\delta}{a_2^2+a_3} > 0. \tag{40}$$

Four fixed points (38) are found to be elliptic, while the other four fixed points (39) are hyperbolic. Therefore, if the conditions (40) are satisfied, resonance structure in this case consists of a chain of four resonant islands. In [10], such a chain of resonant islands is shown for the case $a_2 = 0$. Following the same procedure as in the previous subsection, we derive two symplectic RG maps.

$$\begin{aligned} A'_{n+1} &= A'_n + \varepsilon^2 f_4(A'_{n+1}, A''_n), \\ A''_{n+1} &= A''_n + \varepsilon^2 g_4(A'_{n+1}, A''_n), \end{aligned} \tag{41}$$

where $f_4(A', A'') = -(\delta A'' + (a_{3,1} + a_{3,-3})(A'')^3 + (a_{3,1} - 3a_{3,-3})(A')^2 A'')$ and $g_4(A', A'') = \delta A' + (a_{3,1} + a_{3,-3})(A')^3 + (a_{3,1} - 3a_{3,-3})(A'')^2 A'$. This symplectic RG map takes a set of explicit difference eqations in the case $a_2 = 0$ [10]. Another one preserves a symmery of $2\pi/4$ rotation exactly.

$$\begin{aligned} J_{n+1} &= J_n + \varepsilon^2 F_4(J_{n+1}, \theta_n), \\ \theta_{n+1} &= \theta_n + \varepsilon^2 G_4(J_{n+1}, \theta_n), \end{aligned} \tag{42}$$

where $F_4(J, \theta) = 2a_{3,-3} J^2 \sin(4\theta)$ and $G_4(J, \theta) = \delta + a_{3,1} J + a_{3,-3} J \cos(4\theta)$.

3.3 Resonant Frequency : $2\pi/5$

When the resonant frequency ω_0 is $2\pi/5$, the small resonance detuning is chosen to be of the order ε^2. Then, the similar analysis as in previous sections yields the following perturbation solution. While $x_n^{(2)}$ is given by Eq.(28), $x_n^{(3)}$ is obtained as

$$x_n^{(3)} = i((\delta + a_{3,1}|A|^2) A n \exp(i\omega_0 n)) + a_{3,3} A^3 \exp(3i\omega_0 n) + \text{c.c.}, \tag{43}$$

where

$$a_{3,3} = \frac{2a_2 a_{2,2} + a_3}{2(\cos(3\omega_0) - \cos\omega_0)}$$

and the secular part of $x_n^{(4)}$ is determined by

$$\begin{aligned} Lx_n^{(4)} &= ((2a_2 a_{3,3} + a_2 a_{2,2}^2 + 3a_3 a_{2,2} + a_4)\bar{A}^4) \exp(i\omega_0 n) \\ &\quad + 2ia_2(\delta + a_{3,1}|A|^2) A^2 n \exp(2i\omega_0 n) + \text{c.c.} + \text{non-secular terms}, \end{aligned} \tag{44}$$

where a relation $\exp(\pm 4i\omega_0) = \exp(\mp i\omega_0)$ has been used. A secular solution of Eq.(44) yields the secular part of $x_n^{(4)}$:

$$x_n^{(4)} = -i\alpha_4 \bar{A}^4 n \exp(i\omega_0 n) + ia_{4,2}(\delta + a_{3,1}|A|^2) A^2 n \exp(2i\omega_0 n), \tag{45}$$

where

$$\begin{aligned}\alpha_4 &= \frac{2a_2a_{3,3}+a_2a_{2,2}^2+3a_3a_{2,2}+a_4}{2\sin\omega_0},\\ a_{4,2} &= \frac{a_2}{\cos(2\omega_0)-\cos\omega_0}.\end{aligned}$$

All secular terms in Eqs. (43) and (45) are removed by introducing a renormalization transformation.

$$A_n = A + i\varepsilon^2(\delta + a_{3,1}|A|^2)An - i\varepsilon^3\alpha_4\bar{A}^4 n. \tag{46}$$

Then, we finally obtain a secular-free perturbation solution of the original equation (6) up to $\mathcal{O}(\varepsilon^3)$:

$$\begin{aligned}x_n &= \varepsilon(A_n\exp(i\omega_0 n)+\text{c.c.}) + \varepsilon^2((a_{2,2}A_n^2\exp(2i\omega_0 n)+\text{c.c.}) + a_{2,0}|A_n|^2)\\ &\quad +\varepsilon^3(a_{3,3}A_n^3\exp(3i\omega_0 n)+\text{c.c.}),\end{aligned} \tag{47}$$

with a naive RG map for A_n

$$\begin{aligned}A_{n+1} &= A_n + i\varepsilon^2(\delta + a_{3,1}|A_n|^2)A_n - i\varepsilon^3\alpha_4\bar{A}_n^4 &(48)\\ &= (1+\varepsilon^2\mathcal{L}_H+\cdots)A_n. &(49)\end{aligned}$$

A Hamiltonian generating the naive RG map (48) is given as

$$H(A,\bar{A}) = i(\delta|A|^2 + \frac{a_{3,1}}{2}|A|^4 - \varepsilon\frac{\alpha_4}{5}(A^5+\bar{A}^5)). \tag{50}$$

Although fixed points in the phase plane $(A_n, \bar{A}_n)$ are obtained using the naive RG map (48) or the Hamiltonian (50), we derive them through a symplectic RG map. In terms of the action-angle variables (J_n, θ_n), the map (48) is easily made symplectic as

$$\begin{aligned}J_{n+1} &= J_n + \varepsilon^2 F_5(J_{n+1}, \theta_n),\\ \theta_{n+1} &= \theta_n + \varepsilon^2 G_5(J_{n+1}, \theta_n),\end{aligned} \tag{51}$$

where $F_5(J,\theta) = -2\varepsilon\alpha_4 J^{5/2}\sin(5\theta)$ and $G_5(J,\theta) = \delta + a_{3,1}J - \varepsilon\alpha_4 J^{3/2}\cos(5\theta)$. Fixed points of the map (51) are given by

$$\sin(5\theta)=0, \quad \delta + a_{3,1}J - \varepsilon\alpha_4 J^{3/2}\cos(5\theta) = 0, \tag{52}$$

from which we have 10 fixed points

$$J \approx -\frac{\delta}{a_{3,1}} + \varepsilon\frac{\alpha_4}{a_{3,1}}(-\frac{\delta}{a_{3,1}})^{3/2}, \qquad \theta = \frac{2m\pi}{5}, \tag{53}$$

$$J \approx -\frac{\delta}{a_{3,1}} - \varepsilon\frac{\alpha_4}{a_{3,1}}(-\frac{\delta}{a_{3,1}})^{3/2}, \qquad \theta = \frac{(2m+1)\pi}{5}, \tag{54}$$

where $m = 0, 1, 2, 3, 4$ and it is assumed that

$$-\frac{\delta}{a_{3,1}} > 0. \tag{55}$$

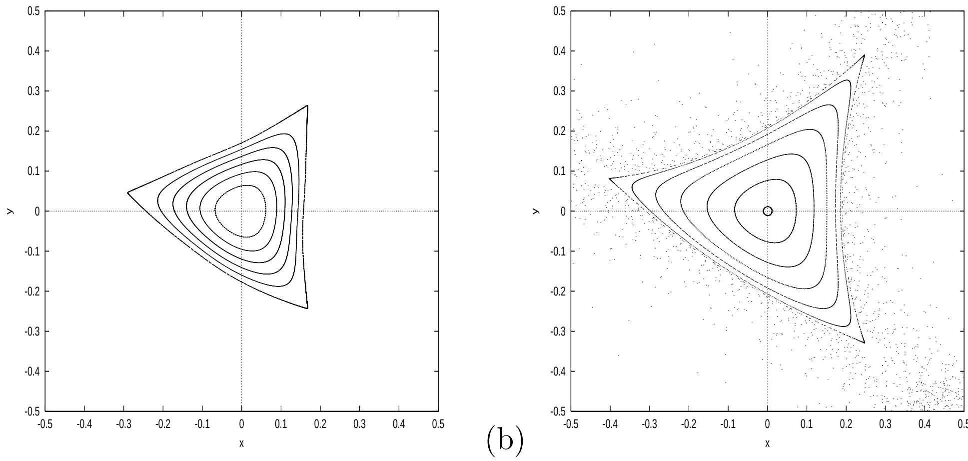
The condition (55) simply states that the resonance detuning δ should be compensated by the nonlinear frequency shift. A set of five fixed points (53) is found to be elliptic while another set (53) is hyperbolic. Therefore, if the resonance detuning is set so that the condition (55) is satisfied, the resonance structure for $2\pi/5$ resonance is composed of a chain of five resonant islands. In the next section, the analysis in the present section is applied to study the resonance structure of the Hénon map.

4 Application to Hénon Map

A map (4) with $a_m = 0$ for $m > 2$ seems to be rather special but has been extensively studied as the Hénon map[7][8]. Let us apply results in the preceding sections to the Hénon map and compare them with numerical calculations.

In the case of $2\pi/3$ resonance near the origin of the phase plane, our analysis indicates that there are only three hyperbolic points of period three and a chain of resonant island does not exist . Numerical calculations of the Hénon map in Fig.1 supports these results of our analysis. In the inner region of a triangle produced by connecting the three hyperbolic periodic points, the present analysis gives a good agreement but is not able to describe a chaotic sea in the outer region of the triangle.

The case of $2\pi/4$ resonance is exceptional in the sense that the former condition, which guarantees existence of elliptic periodic points, in (40) is not satisfied since $a_3 = 0$ for the Hénon map. Therefore, our analysis indicates that there are only four hyperbolic fixed points near the origin of the phase plane. Numerical caluculation confirms existence of the four hyperbolic fixed



(a) (b)
Fig. 1. (a) phase portrait near the $2\pi/3$ resonance obtained from the RG map (26) for $\varepsilon = 0.1$ and $\delta = 0.8$
(b) phase portrait near the $2\pi/3$ resonance derived by a numerical calculation of the Hénon map for the same values of parameters.

points but there also exist four elliptic points far from the origin of the phase space as shown in Fig.2. Our perturbational analysis can not cover a region far from the origin. However, if the third order correction to the RG map is taken into account, a formal estimation shows that four elliptic fixed points exist in a far region, whose distance from the origin is $\mathcal{O}(1/\varepsilon)$.

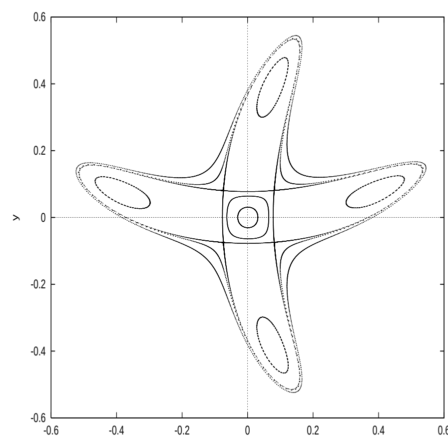
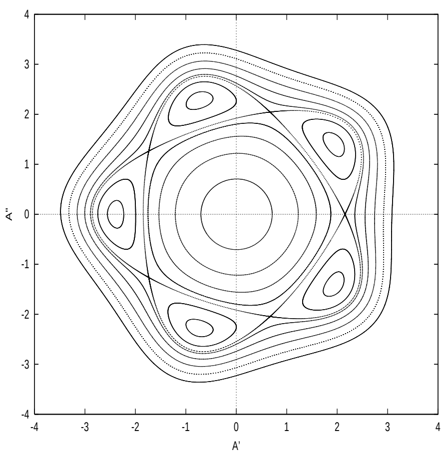


Fig. 2. phase portrait near the $2\pi/4$ resonance calculated from the Hénon map for $\varepsilon = 0.1$ and $\delta = 0.8$

In the case of $2\pi/5$ resonance, we have a beautiful agreement of the analysis and numerical calculations of the Hénon map as shown in Fig.3. While the

phase portrait of the RG map has a symmetry of $2\pi/5$ rotation (Fig.3(a)), the phase structure in the Hénon map does not have such symmetry (Fig.3(c)). This symmetry breaking is realized owing to the higher harmonic terms in the solution (47) as shown in Fig.3(b).

(a)


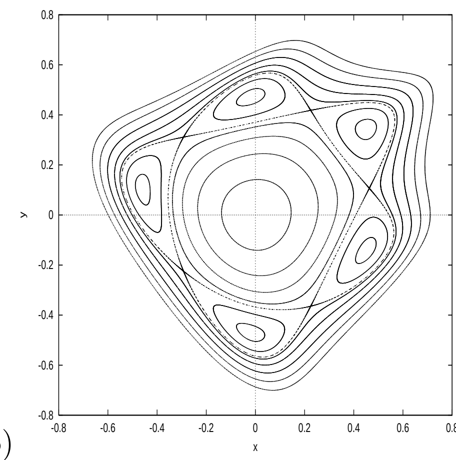
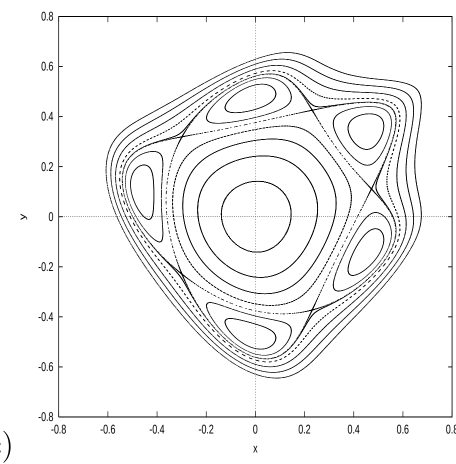
(b)


Fig. 3. (a) phase portrait of the RG map (51) near the $2\pi/5$ resonance for $\varepsilon = 0.1$ and $\delta = 0.8$, where $A' = \sqrt{J}\cos(\theta)$ and $A'' = \sqrt{J}\sin(\theta)$
(b) phase portrait near the $2\pi/5$ resonance in the (x, y) space derived by the RG method
(c) phase portrait near the $2\pi/5$ resonance calculated from the Hénon map

(c)


5 Conclusion

A symplecticity-preserving RG analysis is carried out to study resonance structures near an elliptic fixed point of a proto-type symplectic map in two dimensions. Symplectic RG maps are constructed as symplectic integrators to the underlying Hamilton flows. Fixed points of the RG map correspond

to periodic points of the original map, which determine resonance structures such as a chain of resonant islands. It should be emphasized that we only need to analyze a distribution of fixed points of the RG map in order to determine a topological structure of the resonance region. For the resonant frequency $\omega_0 = 2\pi/k \quad (k = 4, 5)$, the RG map has k hyperbolic fixed points and k elliptic fixed points around the origin with the same angle spacing, while the case $k = 3$ is special because the RG map has only three hyperbolic fixed points. Thus, we conclude that a chain of k resonant islands constitutes the resonance structures for $k = 4, 5$ but such a chain does not exists near the origin for $k = 3$. Although we did not perform analysis for $k \geq 6$ in detail, we are sure that the same conclusion as in the case $k = 4, 5$ is valid in general. In addition to the exisitence of fixed points, the RG map has a significant symmetry property that it is invariant with respect to ω_0 rotation around the origin, although the original proto-type symplectic map does not posses such a symmetry. This rotational symmetry is broken due to the presence of higher hamonics terms and the symmetric RG map can reproduce asymmetric resonance structures in the original phase space.

The above analytical theory for the proto-type symplectic map are checked by numerical caluculations of the Hénon map. The result of numerical calculations is shown to give a good agreement with the theoretical result, especially for the case $k = 5$.

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